

# Mean-field conditions for percolation on finite graphs

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ABSTRACT. Let  $\{G_n\}$  be a sequence of finite transitive graphs with vertex degree  $d = d(n)$  and  $|G_n| = n$ . Denote by  $\mathbf{p}^t(v, v)$  the return probability after  $t$  steps of the *non-backtracking* random walk on  $G_n$ . We show that if  $\mathbf{p}^t(v, v)$  has *quasi-random* properties, then critical bond-percolation on  $G_n$  behaves as it would on a random graph. More precisely, if

$$\limsup_n n^{1/3} \sum_{t=1}^{n^{1/3}} t \mathbf{p}^t(v, v) < \infty,$$

then the size of the largest component in  $p$ -bond-percolation with  $p = \frac{1+O(n^{-1/3})}{d-1}$  is roughly  $n^{2/3}$ . In Physics jargon, this condition implies that there exists a scaling window with a *mean-field* width of  $n^{-1/3}$  around the critical probability  $p_c = \frac{1}{d-1}$ .

A consequence of our theorems is that if  $\{G_n\}$  is a transitive expander family with girth at least  $(\frac{2}{3} + \epsilon) \log_{d-1} n$  then  $\{G_n\}$  has the above scaling window around  $p_c = \frac{1}{d-1}$ . In particular, bond-percolation on the celebrated Ramanujan graph constructed by Lubotzky, Phillips and Sarnak [21] has the above scaling window. This provides the first examples of *quasi-random* graphs behaving like random graphs with respect to critical bond-percolation.

## 1. Introduction

**1.1. Background.** Let  $G$  be a graph and  $p \in [0, 1]$ . Write  $G_p$  for the graph obtained from  $G$  by performing  $p$ -bond-percolation on  $G$ , that is, delete each edge with probability  $1 - p$  and retain it with probability  $p$ , independently for all edges. Denote by  $\mathcal{C}_1$  the largest connected component of  $G_p$ . When  $G$  is the complete graph  $K_n$ , this model is known as the Erdős-Rényi random graph  $G(n, p)$ . Erdős and Rényi [12] discovered at 1960 that when  $p_c = \frac{1}{n}$  the model exhibits a phase transition. Namely, if  $p = \frac{c}{n}$  with  $c < 1$ , then  $|\mathcal{C}_1|$  is of order  $\log n$  and if  $c > 1$ , then  $|\mathcal{C}_1|$  is of order  $n$  and all other components are of logarithmic size.

The study of the random graph around the critical probability (i.e., when  $p \sim \frac{1}{n}$ ) was initiated by Bollobás [6] over twenty years later. He showed that if  $p = \frac{1+O(n^{-1/3})}{n}$  then the size of the largest component in  $G_p$  is roughly  $n^{2/3}$ . He also proved that if

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$p = \frac{1+\epsilon(n)}{n}$  where  $n^{1/3}\epsilon(n) \rightarrow \infty$  then with high probability  $n^{-2/3}|\mathcal{C}_1| \rightarrow \infty$ , and if  $n^{1/3}\epsilon(n) \rightarrow -\infty$  then with high probability  $n^{-2/3}|\mathcal{C}_1| \rightarrow 0$  (Bollobás proved this with some logarithmic corrections which were removed later by Łuczak [19]). In Physics jargon, this phenomenon is frequently called a *scaling window* with *mean-field* width of  $n^{-1/3}$  around  $p_c = \frac{1}{n}$ .

A scaling window of width  $n^{-1/3}$  around  $p_c = \frac{1}{d-1}$  occurs also when  $G$  is a random  $d$ -regular graph on  $n$  vertices where  $d$  is fixed and  $n \rightarrow \infty$  (see [26] and [29]). It is natural to expect, in the spirit of Chung, Graham and Wilson [10], that critical percolation on *deterministic* graphs having *quasi-random* properties will behave the same as random graphs. Up to now, however, no examples of this were known (see section 1.5 for some related results).

**1.2. Mean-field scaling window.** In this paper we show that if the *non-backtracking* random walk (a simple random walk restricted not to traverse the edge it has just visited in the reverse direction. See section 3.1 for a precise definition) behaves on  $G$  as it would on a random graph in some sense, then bond-percolation on  $G$  has the same scaling window as it would on the complete graph (or as it would on a random  $d$ -regular graph). We give a quasi-random condition which guarantees the existence of such a scaling window around  $p_c = \frac{1}{d-1}$ . This condition is defined in terms of return probabilities  $\mathbf{p}^t(v, v)$  of the non-backtracking random walk and should be regarded as a *geometric* condition on  $G$ .

**Theorem 1.** *Let  $\{G_n\}$  be a family of transitive graphs with vertex degree  $d(n) \geq 3$  and assume for simplicity that  $|G_n| = n$ . Let  $p = \frac{1+\lambda n^{-1/3}}{d(n)-1}$  for some fixed  $\lambda \in \mathbb{R}$  and consider the largest component  $\mathcal{C}_1 = \mathcal{C}_1(n)$  of  $p$ -bond-percolation on  $G_n$ . Denote by  $\mathbf{p}^t(v, v)$  the probability that a non-backtracking random walk on  $G_n$  starting at a vertex  $v$  will visit  $v$  at time  $t$ . If*

$$\limsup_n n^{1/3} \sum_{t=1}^{n^{1/3}} t \mathbf{p}^t(v, v) < \infty, \quad (1.1)$$

*then for any  $\epsilon > 0$  there exists  $A = A(\epsilon, \lambda) < \infty$  such that for all  $n$*

$$\mathbf{P}\left(\frac{n^{2/3}}{A} \leq |\mathcal{C}_1| \leq A n^{2/3}\right) \geq 1 - \epsilon.$$

Condition (1.1) holds in various examples. In particular, we have the following consequence for transitive expander graphs. Recall that a sequence of connected graphs  $\{G_n\}$  is called an *expander* family if the largest eigenvalue in absolute value, which is

not  $\pm 1$ , of the transition matrix of the simple random walk on  $G_n$  is strictly smaller than 1, uniformly in  $n$ .

**Theorem 2.** *Let  $\{G_n\}$  be a transitive expander family with vertex degree  $d(n) \geq 3$  and assume that the girth  $g(n)$  (i.e., the length of the shortest cycle) of  $G_n$  satisfies*

$$\limsup_n \left( \frac{1}{d(n)-1} \right)^{\lfloor \frac{g(n)}{2} \rfloor} n^{1/3} \log^2 n < \infty. \quad (1.2)$$

*Then condition (1.1) holds and in particular, if  $\mathcal{C}_1$  is the largest component of  $p$ -bond-percolation on  $G_n$  with  $p = \frac{1+\lambda n^{-1/3}}{d(n)-1}$ , then for any  $\epsilon > 0$  there exists  $A = A(\epsilon, \lambda) < \infty$  such that*

$$\mathbf{P} \left( \frac{n^{2/3}}{A} \leq |\mathcal{C}_1| \leq A n^{2/3} \right) \geq 1 - \epsilon.$$

In particular, bond-percolation on the Ramanujan graphs constructed by Lubotzky, Phillips and Sarnak [21] have a scaling window with mean-field width of  $n^{-1/3}$  (the theorem also applies for constructions of Margulis [23]). These are bounded degree expander graphs with girth approximately  $\frac{4}{3} \log_{d-1} n$ , clearly satisfying the assumption of the theorem with room to spare. Thus, Theorem 2 gives the first class of examples of *quasi-random* graphs having the same behavior as random graphs with respect to critical bond-percolation. We remark that it is proved in [18] that the condition on the girth in Theorem 2 is sharp up to the  $\log^2 n$  factor.

Lower bounds on the size of percolation clusters at criticality is a difficult task in general and is believed to depend heavily on the geometry of the underlying graph. In the case  $p_c = \frac{1}{d-1}$  overcoming this difficulty, not only establishes the existence of a mean-field scaling window, but one immediately reaps additional rewards: other geometric quantities of the largest component, namely the diameter and the mixing time of the simple random walk, assume their mean-field universal values. Indeed, it is proved in [27] that when  $G$  has maximum degree  $d \in [3, n-1]$  and  $p \leq \frac{1+O(n^{-1/3})}{d-1}$  if  $G_p$  typically has components of order  $n^{2/3}$ , then with high probability, these components have diameter of order  $n^{1/3}$  and mixing time of order  $n$ . Hence, the following corollary is an immediate consequence of Theorem 1 and of Theorem 1.2 of [27].

**Corollary 3.** *Assume the setting of Theorem 1 and that condition (1.1) holds. Denote by  $\text{diam}(\mathcal{C}_1)$  and by  $T_{\text{mix}}(\mathcal{C}_1)$  the diameter (maximal graph distance) of  $\mathcal{C}_1$  and the mixing time of the lazy simple random walk on  $\mathcal{C}_1$ , respectively (see [27] for a definition). If  $p = \frac{1+\lambda n^{-1/3}}{d(n)-1}$  then for any  $\epsilon > 0$  there exists  $A = A(\epsilon, \lambda)$  such that*

- $\mathbf{P}\left(\text{diam}(\mathcal{C}_1) \notin [A^{-1}n^{1/3}, An^{1/3}]\right) < \epsilon,$
- $\mathbf{P}\left(T_{\text{mix}}(\mathcal{C}_1) \notin [A^{-1}n, An]\right) < \epsilon.$

**1.3. Outside the scaling window.** In order to show that the critical scaling window is of width  $\Theta(n^{-1/3})$  around  $p_c = \frac{1}{d-1}$  one must show that if  $p = \frac{1+\epsilon(n)}{d(n)-1}$  and  $|n^{1/3}\epsilon(n)| \rightarrow \infty$ , then  $n^{-2/3}|\mathcal{C}_1| \rightarrow \infty$  with probability tending to 1. The lower side of the window is easier to handle. Indeed, in *any*  $d$ -regular graph, if  $p = \frac{1-\epsilon(n)}{d(n)-1}$  with  $|n^{1/3}\epsilon(n)| \rightarrow \infty$  (i.e., the subcritical regime), then  $n^{-2/3}|\mathcal{C}_1| \rightarrow 0$  with probability tending to 1. This is the contents of part (1) of Proposition 1 of [26]. Thus, we only need to take care of the supercritical regime,  $\epsilon(n) > 0$ .

The next theorem shows that a slight variant of condition (1.1) guarantees that  $n^{-2/3}|\mathcal{C}_1| \rightarrow \infty$  with high probability when  $p$  is above the mean-field scaling window.

**Theorem 4.** *Let  $\epsilon(n) > 0$  be a sequence such that  $\epsilon(n) = o(1)$  but  $\epsilon(n)n^{1/3} \rightarrow \infty$ . Take  $p = \frac{1+\epsilon(n)}{d(n)-1}$  and  $r = \epsilon^{-1}[\log(n\epsilon^3) - 3\log\log(n\epsilon^3)]$ . Assume the setting of Theorem 1. We have that if*

$$\limsup_n \epsilon^{-1}r \sum_{t=1}^{2r} [(1+\epsilon)^{t\wedge r} - 1] \mathbf{p}^t(v, v) = 0, \quad (1.3)$$

*then there is some fixed  $\delta > 0$  such that*

$$\mathbf{P}\left(|\mathcal{C}_1| \geq \frac{\delta \epsilon n}{\log^3(n\epsilon^3)}\right) \longrightarrow 1, \quad \text{as } n \rightarrow \infty,$$

*(note that  $\delta \epsilon n \log^{-3}(n\epsilon^3) \gg n^{2/3}$ ).*

Condition (1.3) hold in various examples. Again, we address expander graphs (even though conditions (1.1) and (1.3) do not require the graphs to be connected).

**Theorem 5.** *Under the setting of Theorem 2 we have that condition (1.3) holds. In particular, if  $p = \frac{1+\epsilon(n)}{d(n)-1}$  with  $\epsilon(n) = o(1)$  but  $\epsilon(n)n^{1/3} \rightarrow \infty$ , then there is some fixed  $\delta > 0$  such that*

$$\mathbf{P}\left(|\mathcal{C}_1| \geq \frac{\delta \epsilon n}{\log(n\epsilon^3)}\right) \longrightarrow 1, \quad \text{as } n \rightarrow \infty.$$

**1.4. Products of complete graphs.** In [16], van der Hofstad and Luczak investigate bond-percolation on the  $k$ -dimensional Hamming graph  $H(k, m)$  which has vertex set  $\{0, \dots, m-1\}^k$  and a pair of vertices are connected by an edge if and only if these vertices differ in precisely one coordinate (this is a weak product of  $k$  complete graphs,

each on  $m$  vertices). In these graphs we have that  $n = m^k$  and  $d(n) = k(m - 1)$ . It is shown in [7] and [8], as noted in [16], that bond-percolation on  $H(2, m)$  and  $H(3, m)$  has a scaling window around  $p_c = \frac{1}{d-1}$  of width *at least*  $n^{-1/3}$  (i.e., the consequence of Theorem 1 hold). In [16] the authors provide an upper bound of order  $\log^{1/3}(n)n^{-1/3}$  on the width of the scaling window, i.e., they show that when  $p = \frac{1+\epsilon(n)}{d(n)-1}$  with  $\epsilon(n) = \Omega(\log^{1/3}(n)n^{-1/3})$  then  $n^{-2/3}|\mathcal{C}_1| \rightarrow \infty$  with high probability. In fact, they show that for such  $p$ , the size of the largest component is concentrated around  $2\epsilon(n)n$ .

In the following Theorem we prove that for  $H(2, m)$  and  $H(3, m)$  the scaling window is of width  $\Theta(n^{-1/3})$  around  $p_c = \frac{1}{d(n)-1}$ . We remark that the same conclusion cannot be drawn for  $H(k, m)$  with  $k > 3$  by the results of [18].

**Theorem 6.** *Conditions (1.1) and (1.3) hold for the sequences  $H(2, m)$  and  $H(3, m)$ .*

**1.5. Related results.** It is worth comparing this work to results of a similar flavor given by Borgs, Chayes, van der Hofstad, Slade and Spencer in [7] and [8]. They define a *finite* version of the Aizenman-Newman [2] triangle condition on  $\{G_n\}$  and show that it implies the existence of a scaling window of width *at least*  $n^{-1/3}$  around some  $p_c$ , in which the largest component is of size about  $n^{2/3}$ . They continue in [8] to show that this finite triangle condition holds in a class of graphs including the high-dimensional discrete finite torus  $[m]^d$  (for large fixed  $d$  and  $m \rightarrow \infty$ ) and the hypercube  $\{0, 1\}^m$ .

The important advantage of the results of [7] over Theorem 1 is that it gives a mean-field scaling window around a  $p_c$  which is *not* necessarily  $\frac{1}{d-1}$ . Indeed, in both the tori and the hypercube, at  $p = \frac{1+O(n^{-1/3})}{d-1}$  we have that  $n^{-2/3}|\mathcal{C}_1| \rightarrow 0$  with high probability. Thus, the scaling window given in [7] is at a higher location than  $\frac{1+O(n^{-1/3})}{d-1}$ . However, the advantage of Theorem 1.1 over the results of [7] is that condition (1.1), if it holds, is usually easy to verify while verifying the triangle condition is notoriously difficult (even in the infinite case, see [14]).

Another difference between the two results is that there is no analogue in [7] to Theorem 4. Namely, it is not known whether the finite triangle condition implies that  $n^{-2/3}|\mathcal{C}_1| \rightarrow \infty$  with high probability if  $p = p_c(1 + \epsilon(n))$  with  $\epsilon(n)$  as in Theorem 4. Hence, the finite triangle condition implies only that the scaling window is of size  $\Omega(n^{-1/3})$  but not  $\Theta(n^{-1/3})$ . In our case, conditions (1.1) and (1.3) show that the scaling window assumes the mean-field width of  $\Theta(n^{-1/3})$ .

**1.6. Proof idea.** Our proof is based on the analysis of the BFS (breadth-first-search) exploration process on the percolated graph  $G_p$  starting at a uniformly chosen vertex  $v$ . It is obvious that this process is “dominated” by a BFS process on a percolated

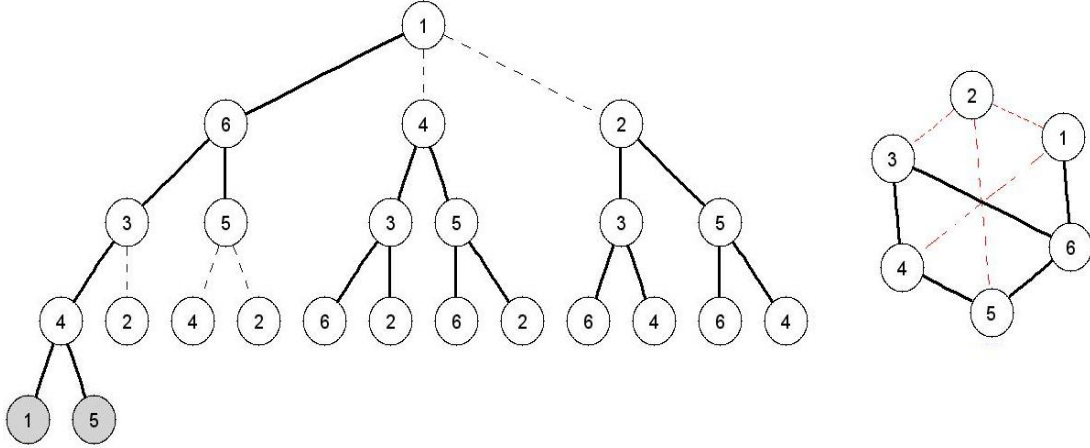


FIGURE 1. On the left, the first levels of  $T_p$  and on the right  $G_p$ . Gray vertices on the left, labeled 5 and 1, are impure.

$d$ -regular infinite tree  $T_p$ . For instance, the size of the component containing  $v$  in  $G_p$  is stochastically dominated by the size of the component containing the root of  $T_p$ . This observation can be used to obtain that the upper bound on  $|\mathcal{C}_1|$  of Theorem 1 (i.e.,  $|\mathcal{C}_1|$  is no more than  $n^{2/3}$ ) holds for *any*  $d$ -regular graph (this is shown in Theorem 1.2 of [27] or Proposition 1 of [26]).

Obtaining a lower bound is much harder and requires additional assumptions on the geometry of the underlying graph (e.g., the triangle condition or condition (1.1)). The starting point of the approach of this paper is to consider how this natural coupling of  $G_p$  with  $T_p$  fails to be sharp. This happens when we encounter a vertex in  $G_p$  which explores *less* than  $d - 1$  of its neighbors. This happens because at least one of its neighbors has been explored before in the BFS process. We consider  $T$  as the covering tree of  $G$  (i.e., to each vertex of  $T$  we associate a vertex of  $G$  such that neighborhoods of vertices of  $G$  are preserved, see section 4.1) and it is clear that the above coupling fails when we explore, this time in  $T_p$ , a vertex for which a vertex with the same label has been discovered before, see figure 1 (its full meaning will be clear in section 4.1). We call such vertices *impure* and a lower bound on the component of  $G_p$  is given by the component of  $T_p$  after removing all impure vertices (and their descendants). This is the contents of Proposition 11.

The next important observation is that in some rough sense, the unique path in  $T$  between an impure vertex and the vertex causing it to be impure (i.e., the vertex with the same label discovered previously) is a random path in  $T$  due to the nature of the BFS exploration process. This path translates to a non-backtracking random

path in  $G$ . This enables us to bound from above the number of impure vertices using our knowledge of the behavior of the non-backtracking random walk on  $G$ .

This technique of estimating component sizes is novel. Previously such estimates were obtained using the *lace expansion* (see [7, 8]) and *sprinkling* (see [9, 16]; sprinkling was introduced in [1]).

**1.7. Organization.** The rest of the paper is organized as follows. Deriving Theorems 2, 5 and 6 from Theorems 1 and 4 is easy and done in the next section. We present some useful preliminaries about non-backtracking random walks and classical bond-percolation on trees in Section 3. Since the statements in Section 3 are easy and classical we advise the reader to treat them as black boxes and continue directly to Section 4, which contains the novel ideas of the proof. This section provides a coupling and a key lemma allowing us to bound from below  $|\mathcal{C}_1|$ . We use the lemmas in Section 4 in a straightforward manner to prove Theorems 1 and 4. We end with some concluding remarks and open problems in Section 6.

## 2. Percolation on expanders

In this short section we derive Theorems 2, 5 and 6 from Theorems 1 and 4. We refer the reader to Section 3.1 for a formal definition of the non-backtracking random walk.

**Proof of Theorems 2 and 5.** The proof simply requires to verify conditions (1.1) and (1.3). A recent result of Alon, Benjamini, Lubetzky and Sodin [3] shows that the non-backtracking random walk on  $d$ -regular, (with  $d \geq 3$ ) non-bipartite expanders mixes faster than the usual simple random walk. It follows from their results that in that case there exists some  $C > 0$  such that for all  $t \geq C \log n$  we have  $\mathbf{p}^t(v, v) \leq \frac{2}{n}$ . If the expander graph  $G$  happens to be bipartite (as in the case of the Lubotzky-Phillips-Sarnak graph [21]), it is clear that  $p^t(v, v) = 0$  if  $t$  is odd. Let  $A$  and  $B$  be the two parts of the graph. We may consider the connected component of  $G^2$  induced on the vertices of  $A$ . One can readily show that this graph is an expander on  $n/2$  vertices, and thus the results of [3] apply. We learn from this discussion that if  $G_n$  is an expander family (either bipartite or non-bipartite), then there exists some fixed  $C > 0$  such that for all  $t \geq C \log n$  we have  $\mathbf{p}^t(v, v) \leq \frac{4}{n}$ .

To handle smaller  $t$ 's we use the girth assumption (in the same way done in [3]). Since the graph spanned on  $\{u : d_G(u, v) \leq \lfloor g/2 \rfloor\}$  is a tree, in order for the walk to return to  $v$  at time  $t$ , it must visit the set  $\{u : d_G(u, v) = \lfloor g/2 \rfloor\}$  at time  $t - \lfloor g/2 \rfloor$  and

then take precisely  $\lfloor g/2 \rfloor$  steps towards  $v$  in the tree. Hence for any  $t \geq g$  we have

$$\mathbf{p}^t(v, v) \leq \left( \frac{1}{d-1} \right)^{\lfloor g/2 \rfloor},$$

and for  $t < g$  it is clear that  $\mathbf{p}^t(v, v) = 0$ . To sum things up, we have

$$\mathbf{p}^t(v, v) \leq \begin{cases} 0, & t < g, \\ \left( \frac{1}{d-1} \right)^{\lfloor g/2 \rfloor}, & g \leq t < C \log n, \\ \frac{4}{n}, & t \geq C \log n. \end{cases}$$

We use this to verify that condition (1.1) holds,

$$\begin{aligned} n^{1/3} \sum_{t=1}^{n^{1/3}} t \mathbf{p}^t(v, v) &\leq n^{1/3} \sum_{t=g}^{C \log n} t \left( \frac{1}{d-1} \right)^{\lfloor g/2 \rfloor} + n^{1/3} \sum_{t=C \log n}^{n^{1/3}} \frac{4t}{n} \\ &\leq C^2 n^{1/3} \log^2 n \left( \frac{1}{d-1} \right)^{\lfloor g/2 \rfloor} + 2. \end{aligned}$$

Thus, assumption (1.2) on the girth implies that condition (1.1) holds. We now verify that condition (1.3) holds,

$$\begin{aligned} \epsilon^{-1} r \sum_{t=1}^{2r} [(1+\epsilon)^{t \wedge r} - 1] \mathbf{p}^t(v, v) &\leq \frac{\epsilon^{-1} r}{(d-1)^{\lfloor g/2 \rfloor}} \sum_{t=g}^{C \log n} [(1+\epsilon)^{t \wedge r} - 1] \\ &\quad + \frac{4\epsilon^{-1} r}{n} \sum_{t=C \log n}^{2r} (1+\epsilon)^{t \wedge r}. \end{aligned} \quad (2.1)$$

We estimate the second term on the right hand side with

$$\sum_{t=C \log n}^{2r} (1+\epsilon)^{t \wedge r} \leq (\epsilon^{-1} + r)(1+\epsilon)^r.$$

Recall that  $r = \epsilon^{-1}[\log(n\epsilon^3) - 3 \log \log(n\epsilon^3)]$ , hence the second term on the right hand side of (2.1) is of order

$$\epsilon^{-1} n^{-1} r^2 (1+\epsilon)^r = O(\log^{-1}(n\epsilon^3)) = o(1),$$

by our assumption on  $\epsilon$ . To estimate the first term on the right hand side of (2.1), note that there exists  $C_2 > 0$  such that for all  $t \leq \epsilon^{-1}$  we have  $(1+\epsilon)^t \leq 1 + C_2 \epsilon t$ . Thus, in the case that  $\epsilon^{-1} \geq C \log(n)$  we estimate this term by

$$\frac{\epsilon^{-1} r}{(d-1)^{\lfloor g/2 \rfloor}} \sum_{t=g}^{C \log n} [(1+\epsilon)^{t \wedge r} - 1] \leq \frac{O(r \log^2 n)}{(d-1)^{\lfloor g/2 \rfloor}} = o(1),$$



by (1.2) (note that  $r = o(n^{1/3})$ ). If  $\epsilon^{-1} \leq C \log(n)$  we estimate

$$\frac{\epsilon^{-1}r}{(d-1)^{\lfloor g/2 \rfloor}} \sum_{t=g}^{C \log n} [(1+\epsilon)^{t \wedge r} - 1] \leq \frac{\epsilon^{-2}r(1+\epsilon)^{C \log n}}{(d-1)^{\lfloor g/2 \rfloor}} \leq \frac{\epsilon^{-3} \log(n\epsilon^3)n^{C\epsilon}}{(d-1)^{\lfloor g/2 \rfloor}} = o(1),$$

by our assumption on  $\epsilon(n)$  and (1.2).  $\square$

**Proof of Theorem 6.** The graphs  $H(2, m)$  and  $H(3, m)$  are expanders with girth 3 (i.e., they have triangles). Assumption (1.2) can be seen to hold for  $H(2, m)$ , however, it does not hold for  $H(3, m)$  because of the  $\log^2 n$  term appearing in (1.2). Therefore, we need to prove that conditions (1.1) and (1.3) hold for  $H(3, m)$ , and to that aim we estimate more carefully  $\mathbf{p}^t(v, v)$ . Take  $v = (0, 0, 0)$  and define the following subsets of the vertex set of  $H(3, m)$ ,

$$A_1 = \left\{ (i, j, k) \in H(3, m) : i = j = 0 \text{ and } k \neq 0 \right\},$$

$$A_2 = \left\{ (i, j, k) \in H(3, m) : i = k = 0 \text{ and } j \neq 0 \right\},$$

$$A_3 = \left\{ (i, j, k) \in H(3, m) : j = k = 0 \text{ and } i \neq 0 \right\}.$$

Observe that in order for the non-backtracking random walk to return to  $v$  at time  $t$ , it must be in  $A_1 \cup A_2 \cup A_3$  at time  $t-1$ . Hence,

$$\mathbf{p}^t(v, v) \leq \frac{1}{d(n)-1} \mathbf{p}^{t-1}(v, A_1 \cup A_2 \cup A_3),$$

where  $\mathbf{p}^{t-1}(v, A)$  is the probability that the walk visits  $A$  at time  $t-1$ . Let  $\{X_t\}$  be the non-backtracking random walk. We have

$$\begin{aligned} \mathbf{p}^{t-1}(v, A_1) &\leq \mathbf{P}\left(X_j \in A_1 \cup \{v\} \text{ for all } j \leq t-1\right) \\ &+ \mathbf{P}\left(\exists j \leq t-1 \text{ with } X_j \notin A_1 \cup \{v\} \text{ and } X_{j+1} \in A_1\right). \end{aligned}$$

It is clear that the probability of the first event is  $(\frac{2}{3})^{t-1}$  because it requires that the walk does not walk on coordinates 2 and 3 for  $t-1$  steps. We can bound the probability of the second event above by  $\frac{t}{d(n)-1}$ , since the probability of a particular  $j$  having  $X_j \notin A_1 \cup \{v\}$  and  $X_{j+1} \in A_1$  is bounded above by  $\frac{1}{d(n)-1}$  (because it needs to walk to 0 at the first coordinate). We deduce by all this that

$$\mathbf{p}^t(v, v) \leq \begin{cases} 0, & t < 3, \\ \frac{3}{d(n)-1} \left( \left(\frac{2}{3}\right)^{t-1} + \frac{t}{d(n)-1} \right), & g \leq t < C \log n, \\ \frac{2}{n}, & t \geq C \log n. \end{cases}$$

We use this to verify that condition (1.1) holds,

$$n^{1/3} \sum_{t=1}^{n^{1/3}} t \mathbf{p}^t(v, v) \leq \frac{3n^{1/3}}{d(n)-1} \sum_{t=3}^{C \log n} t (2/3)^{t-1} + \frac{3n^{1/3}}{(d(n)-1)^2} \sum_{t=3}^{C \log n} t^2 + n^{1/3} \sum_{t=C \log n}^{n^{1/3}} \frac{2t}{n}.$$

Recalling that  $d(n) = \Theta(n^{1/3})$  shows that condition (1.1) holds. We now verify that condition (1.3) holds,

$$\begin{aligned} \epsilon^{-1} r \sum_{t=1}^{2r} [(1+\epsilon)^{t \wedge r} - 1] \mathbf{p}^t(v, v) &\leq \frac{3\epsilon^{-1} r}{d(n)-1} \sum_{t=3}^{C \log n} [(1+\epsilon)^{t \wedge r} - 1] \left(\frac{2}{3}\right)^{t-1} \\ &+ \frac{3\epsilon^{-1} r}{(d(n)-1)^2} \sum_{t=3}^{C \log n} [(1+\epsilon)^{t \wedge r} - 1] t \\ &+ \frac{\epsilon^{-1} r}{n} \sum_{t=C \log n}^{2r} (1+\epsilon)^{t \wedge r}. \end{aligned}$$

The last term on the right hand side tends to 0 as in (2.1). To estimate the two other terms, assume first  $\epsilon^{-1}(n) \geq C \log n$ , then as before  $(1+\epsilon)^t \leq 1 + C_2 \epsilon t$  and we have

$$\frac{3\epsilon^{-1} r}{d(n)-1} \sum_{t=3}^{C \log n} [(1+\epsilon)^{t \wedge r} - 1] \left(\frac{2}{3}\right)^{t-1} \leq \frac{O(r)}{d(n)-1} \sum_{t=3}^{\infty} t \left(\frac{2}{3}\right)^{t-1} = o(1),$$

since  $r = o(n^{1/3})$ . Similarly,

$$\frac{3\epsilon^{-1} r}{(d(n)-1)^2} \sum_{t=3}^{C \log n} [(1+\epsilon)^{t \wedge r} - 1] t \leq \frac{O(r \log^3 n)}{(d(n)-1)^2} = o(1).$$

The case  $\epsilon^{-1}(n) \leq C \log n$  is handled similarly and we conclude that conditions (1.1) and (1.3) indeed hold.  $\square$

### 3. Preliminaries

**3.1. The non-backtracking random walk.** The non-backtracking random walk is a simple random walk on a graph not allowed to traverse back on an edge it has just walked on. Formally, the non-backtracking random walk on an undirected graph  $G = (V, E)$ , starting from a vertex  $x \in V$ , is a Markov chain  $\{X_t\}$  with transition matrix  $\mathbf{P}^x$  on the state space of *directed* edges

$$\vec{E} = \left\{ (x, y) : \{x, y\} \in E \right\}.$$

If  $X_t = (x, y)$  we write  $X_t^{(1)} = x$  and  $X_t^{(2)} = y$ . Also, for notational convenience, we write  $\mathbf{P}_{(x,y)}(\cdot)$  for  $\mathbf{P}^x(\cdot \mid X_0 = (x, w))$ , and  $\mathbf{p}^t(x, y)$  for  $\mathbf{P}^x(X_t^{(2)} = y)$ . The non-backtracking walk starting from a vertex  $x$  has initial state given by

$$\mathbf{P}^x(X_0 = (x, y)) = \mathbf{1}_{\{(x,y) \in \vec{E}\}} \frac{1}{\deg(x)},$$

and transition probabilities given by

$$\mathbf{P}_{(x,y)}^x(X_1 = (y, z)) = \mathbf{1}_{\{(y,z) \in \vec{E}, z \neq x\}} \frac{1}{\deg(y) - 1},$$

where we write  $\deg(x)$  for the degree of  $x$  in  $G$ . The following Lemma is the analogue of the statement  $\mathbf{p}^{t+t'}(x, x) = \sum_y \mathbf{p}^t(x, y) \mathbf{p}^{t'}(y, x)$  which holds for the simple random walk.

**Lemma 7.** *Let  $G = (V, E)$  be a transitive graph with vertex degree  $d$  and  $v \in V$  be an arbitrary vertex. Denote  $v$ 's neighbors in  $G$  by  $v_1, \dots, v_d$ . Then for any two positive integer  $t, t'$  we have*

$$\sum_{y \in V} \sum_{\substack{i,j=1 \\ i \neq j}}^d \mathbf{P}_{(v,v_i)}(X_t^{(2)} = y) \mathbf{P}_{(v,v_j)}(X_{t'}^{(2)} = y) = d(d-1) \mathbf{p}^{t+t'+1}(v, v).$$

**Proof.** We expand  $\mathbf{p}^{t+t'+1}(v, v)$  by conditioning on the location of the chain at time  $t+1$ . The Markov property gives,

$$\mathbf{p}^{t+t'+1}(v, v) = \sum_{e=(x,y)} \mathbf{P}_{(x,y)}(X_{t'}^{(2)} = v) \mathbf{P}^v(X_{t+1} = (x, y)). \quad (3.1)$$

For a vertex  $x \in V$  write  $x_1, \dots, x_d$  for the neighbors of  $x$  in  $G$ . We use

$$\mathbf{P}^v(X_{t+1} = (x, y)) = \frac{1}{d-1} \sum_{\substack{j=1 \\ x_j \neq y}}^d \mathbf{P}^v(X_t = (x_j, x)),$$

to rewrite (3.1) as

$$\mathbf{p}^{t+t'+1}(v, v) = \frac{1}{d-1} \sum_{x \in V} \sum_{\substack{i,j=1 \\ i \neq j}}^d \mathbf{P}_{(x,x_i)}(X_{t'}^{(2)} = v) \mathbf{P}^v(X_t = (x_j, x)). \quad (3.2)$$

Write  $N(v, x_j, x, t)$  for the number of non-backtracking paths of length  $t$  from  $v$  to  $x$  such that the directed edge visited at time  $t$  (the last edge) is  $(x_j, x)$ . We have  $\mathbf{P}^v(X_t = (x_j, x)) = N(v, x_j, x, t)/d(d-1)^{t-1}$ . By traversing the paths in reverse we learn that  $N(v, x_j, x, t)$  is the number of non-backtracking paths of length  $t$ , starting with the edge

$(x, x_j)$  and ending in  $v$ . Hence we have  $\mathbf{P}_{(x, x_j)}(X_t^{(2)} = v) = N(v, x_j, x, t)/(d-1)^{t-1}$ . We deduce that

$$\mathbf{P}^v(X_t = (x_j, x)) = \frac{1}{d} \mathbf{P}_{(x, x_j)}(X_t^{(2)} = v).$$

We put this into (3.2) and get

$$\mathbf{P}^{t+t'+1}(v, v) = \frac{1}{d(d-1)} \sum_{x \in V} \sum_{\substack{i, j=1 \\ i \neq j}}^d \mathbf{P}_{(x, x_i)}(X_{t'}^{(2)} = v) \mathbf{P}_{(x, x_j)}(X_t^{(2)} = v).$$

We sum on  $v \in V$  and divide by  $n$  both sides to get

$$\frac{1}{n} \sum_{v \in V} \mathbf{P}^{t+t'+1}(v, v) = \frac{1}{d(d-1)n} \sum_{x \in V} \sum_{v \in V} \sum_{\substack{i, j=1 \\ i \neq j}}^d \mathbf{P}_{(x, x_i)}(X_{t'}^{(2)} = v) \mathbf{P}_{(x, x_j)}(X_t^{(2)} = v) \quad (3.3)$$

Observe that because  $G$  is transitive we have  $\frac{1}{n} \sum_{v \in V} \mathbf{P}^{t+t'+1}(v, v) = \mathbf{P}^{t+t'+1}(v, v)$ . Also due to transitivity we have that for any  $x \in V$  the sum

$$\sum_{v \in V} \sum_{\substack{i, j=1 \\ i \neq j}}^d \mathbf{P}_{(x, x_i)}(X_{t'}^{(2)} = v) \mathbf{P}_{(x, x_j)}(X_t^{(2)} = v),$$

evaluates to the same number. The assertion of the lemma then follows from (3.3).  $\square$

**3.2. Critical percolation on trees.** Let  $T$  be an infinite  $d$ -regular tree rooted at a vertex  $\rho$ . For a vertex  $u \in T$  we write  $|u|$  for the distance of  $u$  from  $\rho$  (i.e., the number of edges in the path between  $u$  and  $\rho$ ). We say  $w$  is an *ancestor* of  $u$  (or  $u$  is a *descendant* of  $w$ ) if  $w$  belongs to the unique path connecting  $\rho$  and  $u$ . For a vertex  $w \in T$  with  $w \neq \rho$ , denote by  $w^-$  the immediate ancestor (the father) of  $w$  in  $T$ . We also use the notation  $u \wedge w$  for the common ancestor of  $u$  and  $w$  having maximal distance from  $\rho$ .

Let  $p \in [0, 1]$  and consider  $p$ -bond percolation on  $T$ . Denote the resulting subgraph by  $T_p$ . For two vertices  $u, w \in T$ , we denote by  $\{u \leftrightarrow w\}$  the event that  $u$  is connected to  $w$  in  $T_p$ . For an integer  $r > 0$  we write  $H_r$  for the  $r$ -th level of  $T_p$ , i.e.,

$$H_r = \left| \left\{ w \in T : |w| = r, w \leftrightarrow \rho \right\} \right|.$$

We will frequently use the following two lemmas dealing with percolation on  $T$  with  $p = \frac{1+\epsilon}{d-1}$ . The two lemmas estimate the same quantities, but since percolation at this regime changes drastically with the sign of  $\epsilon$ , the estimates given in them are different.

**Lemma 8.** *Let  $\epsilon \in (0, 1/2)$  and put  $p = \frac{1+\epsilon}{d-1}$ . For any integer  $r > 0$  we have*

$$\mathbb{E}H_r^2 = O\left(\epsilon^{-1}(1+\epsilon)^{2r}\right), \quad (3.4)$$

$$\frac{\epsilon}{2} \leq \mathbf{P}\left(H_r > 0\right) \leq 12\epsilon(1 - e^{-\epsilon r/2})^{-1}, \quad (3.5)$$

$$\mathbb{E}\left[\left(\sum_{k=r/2}^r H_k\right)^2 \mid H_{r/2} > 0\right] = O(\epsilon^{-4}(1+\epsilon)^{2r}). \quad (3.6)$$

**Lemma 9.** *Let  $\epsilon \in (0, 1/2)$  and put  $p = \frac{1-\epsilon}{d-1}$ . For any integer  $r > 0$  we have*

$$\mathbb{E}H_r^2 = O\left(\epsilon^{-1}(1-\epsilon)^r\right), \quad (3.7)$$

$$\frac{\epsilon(1-\epsilon)^r}{2} \leq \mathbf{P}\left(H_r > 0\right) \leq \frac{12}{r}, \quad (3.8)$$

$$\mathbb{E}\left[\left(\sum_{k=r/2}^r H_k\right)^2 \mid H_{r/2} > 0\right] = O(\epsilon^{-3}r). \quad (3.9)$$

For the proof of (3.5) and (3.8) we will use the following result due to Lyons [20].

**Lemma 10** (Theorem 2.1 of [20]). *Assign each edge  $e$  from level  $b-1$  to level  $b$  of  $T$  the edge resistance  $r_e = \frac{1-p}{p^b}$ . Let  $\mathcal{R}_k$  be the effective resistance from the root to level  $k$  of  $T$ . Then*

$$\frac{1}{1 + \mathcal{R}_k} \leq \mathbf{P}(H_k > 0) \leq \frac{2}{1 + \mathcal{R}_k}.$$

**Proof of Lemma 8.** We have

$$\mathbb{E}H_r^2 = \sum_{j=0}^r f(j)p^{2r-j},$$

where

$$f(j) = \left| \left\{ \{w_1, w_2\} : |w_1| = |w_2| = r, |w_1 \wedge w_2| = j \right\} \right|.$$

We have  $f(0) = d(d-1)^{2r-1}/2$  and  $f(r) = d(d-1)^{r-1}$  and  $f(j) = d(d-2)(d-1)^{2r-j-2}/2$  for  $1 \leq j \leq r-1$ . We get

$$\begin{aligned} \mathbb{E}H_r^2 &= \frac{d(1+\epsilon)^{2r}}{2(d-1)} + \frac{d(1+\epsilon)^r}{d-1} + \frac{d(d-2)(1+\epsilon)^{2r}}{2(d-1)^2} \sum_{j=1}^{r-1} (1+\epsilon)^{-j} \\ &= O\left(\frac{(1+\epsilon)^{2r}}{\epsilon}\right), \end{aligned} \quad (3.10)$$

which finishes the proof of (3.4). To prove (3.5), in the setting of Lemma 10, we have that the effective resistance  $\mathcal{R}_r$  from  $\rho$  to level  $r$  of  $T$  satisfies (see [28], Example 8.3)

$$\begin{aligned}\mathcal{R}_r &= \sum_{i=1}^r \frac{(1-p)p^{-i}}{d(d-1)^{i-1}} = \sum_{i=1}^r \frac{\frac{d-2-\epsilon}{d-1}(d-1)^i(1+\epsilon)^{-i}}{d(d-1)^{i-1}} \\ &= \frac{d-2-\epsilon}{d} \left[ \epsilon^{-1}(1 - (1+\epsilon)^{-r}) \right].\end{aligned}\tag{3.11}$$

We bound the last term using the estimates

$$\frac{1}{6} \leq \frac{d-2-\epsilon}{d} \leq 1, \quad 0 \leq (1+\epsilon)^{-r} \leq e^{-\epsilon r/2},$$

which are valid for  $d \geq 3$  and  $\epsilon \in (0, 1/2)$  (since  $1+x \geq e^{x/2}$  for  $x \in [0, 1/2]$ ). We get that

$$\frac{\epsilon^{-1}(1 - e^{-\epsilon r/2})}{6} \leq \mathcal{R}_r \leq \epsilon^{-1},$$

which together with Lemma 10 yields (3.5). To prove (3.6) note that for any  $k_1 \geq k_2 \geq r/2$  we have

$$\mathbb{E}[H_{k_1}H_{k_2} \mid H_{r/2} > 0] = \frac{\mathbb{E}H_{k_1}H_{k_2}}{\mathbf{P}(H_{r/2} > 0)}.$$

Thus,

$$\mathbb{E}\left[\left(\sum_{k=r/2}^r H_k\right)^2 \mid H_{r/2} > 0\right] = \frac{2}{\mathbf{P}(H_{r/2} > 0)} \sum_{k_1 \geq k_2 \geq r/2}^r \mathbb{E}H_{k_1}H_{k_2}.\tag{3.12}$$

For any  $k_1 \geq k_2$  we have

$$\mathbb{E}[H_{k_1}H_{k_2}] = \mathbb{E}[H_{k_2}\mathbb{E}[H_{k_1} \mid H_{k_2}]] = (1+\epsilon)^{k_1-k_2}\mathbb{E}[H_{k_2}^2] = O\left(\epsilon^{-1}(1+\epsilon)^{k_1+k_2}\right),$$

by (3.4). We put this into (3.12) and use the lower bound of (3.5) to get that

$$\mathbb{E}\left[\left(\sum_{k=r/2}^r H_k\right)^2 \mid H_{r/2} > 0\right] \leq O(\epsilon^{-2}) \sum_{k_1=r/2}^r \sum_{k_2=r/2}^{k_1} (1+\epsilon)^{k_1+k_2} = O(\epsilon^{-4}(1+\epsilon)^{2r}),$$

finishing the proof of (3.6).  $\square$

**Proof of Lemma 9.** We proceed in the same manner as in the previous lemma. Indeed, equality (3.10) and an immediate calculation imply (3.7). The equality (3.11) on  $\mathcal{R}_r$  holds and we get

$$\mathcal{R}_r = \frac{d-2+\epsilon}{d} \left[ \epsilon^{-1}((1-\epsilon)^{-r} - 1) \right].$$

We use  $(1-\epsilon)^{-r} - 1 \geq \epsilon r$  and similar estimates to previous lemma to get

$$\frac{r}{6} \leq \mathcal{R}_r \leq \epsilon^{-1}(1-\epsilon)^{-r},$$

which together with Lemma 10 yields (3.8). As before, using (3.7) we get that for any  $k_1 \geq k_2$

$$\mathbb{E}H_{k_1}H_{k_2} = (1 - \epsilon)^{k_1 - k_2} \mathbb{E}H_{k_2}^2 = O\left(\epsilon^{-1}(1 - \epsilon)^{k_1}\right).$$

We put this into (3.12) and use (3.8) to estimate

$$\mathbb{E}\left[\left(\sum_{k=r/2}^r H_k\right)^2 \mid H_{r/2} > 0\right] \leq O\left(\frac{\epsilon^{-1}}{(1 - \epsilon)^{r/2}}\right) \sum_{k_1=r/2}^r \sum_{k_2=r/2}^{k_1} \epsilon^{-1}(1 - \epsilon)^{k_1} = O(\epsilon^{-3}r),$$

concluding our proof.  $\square$

#### 4. A lower bound on component size

**4.1. A coupling.** Let  $G$  be a transitive graph on  $n$  vertices with vertex degree  $d$  and let  $v \in G$  be an arbitrary vertex of  $G$ . We denote by  $V(G)$  and  $E(G)$  the vertex and edge set of  $G$ , respectively. The *covering tree* of  $G$  rooted at  $v$  is a pair  $(T, \mathcal{L})$  where  $T$  is an infinite  $d$ -regular tree rooted at a vertex  $\rho$  and  $\mathcal{L}$  is a function  $\mathcal{L} : T \rightarrow V(G)$  satisfying:

- (1)  $\mathcal{L}(\rho) = v$ , and  $\mathcal{L}(\rho_i) = v_i$  for  $i \in \{1, \dots, d\}$ , where  $v_1, \dots, v_d$  are the neighbors of  $v$  in  $G$  and  $\rho_1, \dots, \rho_d$  are the children of  $\rho$  in  $T$ .
- (2) For  $w \in T \setminus \{\rho\}$  we have

$$\left\{\mathcal{L}(w_1), \dots, \mathcal{L}(w_{d-1})\right\} = \left\{v : (\mathcal{L}(w), v) \in E(G) \text{ and } v \neq \mathcal{L}(w^-)\right\},$$

where  $w_1, \dots, w_{d-1}$  are the  $d-1$  children of  $w$  and  $w^-$  is the immediate ancestor of  $w$ .

We regard  $\mathcal{L}$  as a labelling of the vertices of the infinite tree  $T$  with labels taking values in  $V(G)$ . It is clear that, up to the choice of the arbitrary mapping between the two sets in requirement (2) of the definition, the covering tree of  $G$  rooted at  $v$  is unique. For the following definitions we assume  $(T, \mathcal{L})$  is a covering tree of  $G$  rooted at  $v$  and  $T_p$  is the subgraph of  $T$  obtained by performing  $p$ -bond-percolation on  $T$  with  $p \in [0, 1]$ .

**Definition 1.** A vertex  $w \in T$  is called **impure** if there exists a vertex  $u \in T$  satisfying

$$(1) |u| \leq |w|, \quad (2) \mathcal{L}(u) = \mathcal{L}(w), \quad (3) u \leftrightarrow u \wedge w.$$

See Figure 1 for an example. A vertex  $w \in T$  is called **pure** if it is not impure. We call a vertex  $w \in T$  **path-pure** if every vertex on the unique path between  $w$  and  $\rho$  is pure.

**Definition 2.** For a vertex set  $A \subset G$ , we say  $w \in T$  is  $A$ -**free** if every vertex  $u$  on the unique path between  $w$  and  $\rho$  has  $\mathcal{L}(u) \notin A$ .

For a subset of vertices  $A \subset G$  and two vertices  $u, v \notin A$  we write  $d_p^A(u, v)$  for the graph distance between  $u$  and  $v$  in  $G_p \setminus A$  (i.e., in the graph obtained from  $G_p$  by removing the vertices of  $A$  and all the edges adjacent to  $A$ ). We denote

$$B_p^A(v, r) = \{u : d_p^A(v, u) \leq r\},$$

$$\partial B_p^A(v, r) = \{u : d_p^A(v, u) = r\}.$$

The following random variable plays a key role in the proofs. For an integer  $r \geq 0$  and a vertex subset  $A \subset G$  we define  $X_r^A$  by

$$X_r^A = \left| \left\{ w \in T : |w| = r, w \leftrightarrow \rho, w \text{ is path-pure and } w \text{ is } A\text{-free} \right\} \right|.$$

**Proposition 11** (Coupling). *Let  $G$  be a  $d$ -regular graph and  $(T, \mathcal{L})$  its covering tree. Recall the definition of  $H_r$  from Section 3.2. For any  $p \in [0, 1]$  and any subset  $A \subset G$  there exists a coupling of  $G_p$  and  $T_p$  such that*

$$X_r^A \leq |\partial B_p^A(v, r)| \leq H_r. \quad (4.1)$$

**Proof.** We recall a *breadth first search* process which explores  $B_p^A(v, r)$  in the spirit of Martin-Löf [24] and Karp [17]. In this exploration process, vertices of  $G$  can be either *explored*, *active* or *neutral*. The active vertices are ordered in a queue which initially contains only  $v$  while the rest of the vertices are neutral. We define a height function  $h : G \setminus A \rightarrow \mathbb{Z}^+ \cup \{\infty\}$  which is updated as the exploration process runs. Initially we have  $h(v) = 0$  and  $h(u) = \infty$  for any  $u \neq v$ . At step  $t$  of the process, we take out the first active vertex  $x_t$  from the queue. If  $h(x_t) = r$  we mark it explored, and proceed with the next step of the process. If  $h(x_t) < r$ , we “explore” the edges between itself and its neutral neighbors in  $G \setminus A$ . That is, if  $x_t$  has  $k$  neutral neighbors in  $G \setminus A$ , we examine these  $k$  edges and check whether they are open or closed (where open or closed edges correspond to retained or deleted edges in the percolation, respectively). For each open edge, we mark the corresponding neutral neighbor  $u$  as active, put  $h(u) = h(x_t) + 1$  and add  $u$  to the end of the queue. We then mark  $x_t$  as explored and proceed with the next step of the process. The process ends when the queue empties. We make two important observations. First, the BFS tree structure of this



process guarantees that we will never have in the queue two vertices which have height difference strictly larger than 1. Secondly, at the end of this process we have

$$\partial B_p^A(v, r) = \left\{ x : h(x) = r \right\}.$$

We couple this exploration process, in a natural way, with percolation on  $T$ . Each edge explored in the exploration process will correspond to precisely one edge in the tree and these two edges are coupled such that they are open or closed together; the rest of the edges in the tree will be open with probability  $p$  and closed otherwise, independently of all other choices. We will also have a correspondence between the vertices of  $B_p^A(v, r)$  and the vertices of  $T$ . The correspondence (of both edges and vertices) is done in the following recursive manner. Recall that  $v_1, \dots, v_d$  are the neighbors of  $v$  in  $G$  and  $\rho_1, \dots, \rho_d$  are the children of  $\rho$  in  $T$ . Initially, the vertex  $v$  corresponds to  $\rho$  and the edges  $(v, v_i)$  and  $(\rho, \rho_i)$  correspond to each other and are coupled such that they are open or closed together. At step  $t > 1$ , let  $x_t$  be the active vertex explored and let  $y_1, \dots, y_k$  be its *neutral* neighbors in  $G \setminus A$ ; note that  $k \leq d - 1$ . Let  $w_t$  be the vertex corresponding to  $x_t$  in  $T$  and denote by  $u_1, \dots, u_k$  the children of  $w_t$  in  $T$  such that  $\mathcal{L}(w_i) = y_i$  for  $1 \leq i \leq k$ . We now have the edge  $(x_t, y_i)$  and  $(w_t, u_i)$  correspond to each other and we couple such that they are open or closed together, for  $1 \leq i \leq k$ . We also have the graph vertex  $y_i$  correspond to the tree vertex  $u_i$  for each *open* edge  $(x_t, y_i)$ . This finishes the description of our coupling.

Under this coupling, the upper bound of (4.1) is obvious and we need to prove the lower bound. We do this in two steps.

**Claim 1.** If a graph vertex  $y \in G$  corresponds in our coupling to a tree vertex  $u \in T$ , then  $u \leftrightarrow \rho$  and  $h(y) = |u|$  and  $v = \mathcal{L}(u)$ .

**Proof.** Follows easily from the definition of our coupling by induction on  $h(y)$ .

**Claim 2.** If a tree vertex  $w \in T$  is path pure and  $A$ -free and has  $|w| = \ell$  and  $w \leftrightarrow \rho$ , then  $\mathcal{L}(w)$  corresponds in our coupling to  $w$  and for every pure child  $w^+$  of  $w$ , if the tree edge  $(w, w^+)$  is open, then it corresponds in our coupling to  $(\mathcal{L}(w), \mathcal{L}(w^+))$ .

**Proof.** We prove by induction on  $\ell$ . The assertion is obvious for  $\ell = 0$ . Let  $\ell > 0$  and assume  $w$  satisfies the assumptions of the claim. Denote the path from  $\rho$  to  $w$  in  $T$  by  $\rho = w_0, w_1, \dots, w_\ell = w$ . We apply our induction hypothesis on  $w_{\ell-1}$  and deduce that  $\mathcal{L}(w_{\ell-1})$  corresponds to  $w_{\ell-1}$  and hence  $h(\mathcal{L}(w_{\ell-1})) = \ell - 1$  by Claim 1. We also deduce that the tree edge  $(w_{\ell-1}, w_\ell)$  corresponds to the edge  $(\mathcal{L}(w_{\ell-1}), \mathcal{L}(w_\ell))$  whence the edge  $(\mathcal{L}(w_{\ell-1}), \mathcal{L}(w_\ell))$  is open in  $G_p$ . Consider the time  $t$  when  $\mathcal{L}(w_{\ell-1})$  was the active vertex  $x_t$  taken out of the queue. We claim that at that time the graph vertex  $\mathcal{L}(w_\ell)$  was neutral. Assume otherwise, then  $\mathcal{L}(w_\ell)$  is active or explored at time

$t$  and since  $h(\mathcal{L}(w_{\ell-1})) = \ell - 1$  we have that  $h(\mathcal{L}(w_\ell)) \leq \ell$  by our first observation from before. We deduce that  $\mathcal{L}(w_\ell)$  corresponds to some tree vertex  $u$  where  $u \neq w_\ell$  (because this correspondence exists at time  $t$ ). By claim 1 we have that  $u \leftrightarrow \rho$  and  $\mathcal{L}(u) = \mathcal{L}(w_\ell)$  and  $|u| \leq |w_\ell|$ , hence  $w_\ell$  is not pure and we have reached a contradiction.

We learn that  $\mathcal{L}(w_\ell)$  was neutral at time  $t$  and since the edge  $(\mathcal{L}(w_{\ell-1}), \mathcal{L}(w_\ell))$  is open. Thus, it will be examined at time  $t$ , concluding our proof.  $\square$

**Remark.** There can be strict inequality in the lower bound of Proposition 11. Indeed, it may happen that  $w$  is impure because of some vertex  $u$ , but  $u$  is impure itself (or the path between  $u$  to  $u \wedge w$  is not a pure path), and in this case we may see  $\mathcal{L}(w)$  in the exploration process on  $G$ , but not count it in  $X_r^A$ . For example, in Figure 1, if the edge between 5 and 4 in the left side of the tree was open, then both of these 4's would be impure.

**4.2. A key lemma.** The following key lemma utilizes the non-backtracking random walk to provide a lower bound on the expected size of  $B_p^A(v, r)$ . In all the lemmas of the rest of Section 4 we have a transitive graph  $G$  with vertex degree  $d$ . Recall from Section 3.1 that  $\mathbf{p}^t(v, v)$  is the return probability after  $t$  steps of the non-backtracking random walk.

**Lemma 12.** *Let  $v$  be a uniform random vertex of  $G$ . Then for any  $p \in [0, 1]$  and any  $A \subset G$  we have*

$$\mathbb{E}|\partial B_p^A(v, r)| \geq (p(d-1))^r \left[ 1 - \frac{r|A|}{n} - \frac{d}{d-1} \sum_{h=2}^r \sum_{k=2}^h \sum_{j=1}^{k-1} (p(d-1))^{k-j} \mathbf{p}^{h+k-2j-1}(v, v) \right].$$

**Proof.** Let  $(T, \mathcal{L})$  be the covering tree of  $G$  rooted at  $v$  and consider  $T_p$ . By proposition 11 it suffices to prove the estimate of the lemma on  $\mathbb{E}X_r^A$ . To that aim, let  $W_r$  be a uniform random vertex from the set  $\{w \in T : |w| = r\}$  and let  $(W_0, W_1, \dots, W_r)$  denote the random path from  $W_0 = \rho$  to  $W_r$ . For triples of integers  $(j, k, h)$  satisfying  $1 \leq j < k \leq h \leq r$  denote by  $X_{j,k}^{(h)}$  the random variable

$$X_{j,k}^{(h)} = \left| \left\{ u \in T : |u| = k, u \wedge W_h = W_j, \mathcal{L}(u) = \mathcal{L}(W_h), u \leftrightarrow W_j \right\} \right|.$$

This random variable counts the number of vertices of height  $k$  in  $T$  which connect to the path  $(W_0, \dots, W_r)$  at  $W_j$  and make the vertex  $W_h$  impure. Indeed, observe that by definition, if  $X_{j,k}^{(h)} = 0$  for all triples with  $1 \leq j < k \leq h \leq r$  then  $W_r$  is path-pure. We have that

$$\mathbb{E}X_r^A = d(d-1)^{r-1} \mathbf{P}\left(W_r \leftrightarrow \rho, W_r \text{ is path-pure and } W_r \text{ is } A\text{-free}\right).$$

For any instance of  $W_r$ , the random variable  $X_{j,k}^{(h)}$  is determined by percolation on edges which are *not* the edges of the path  $W_0, \dots, W_r$ . Hence  $X_{j,k}^{(h)}$  is independent of the event  $\{W_r \leftrightarrow \rho\}$  for all triples  $(j, k, h)$ . Similarly, the event  $\{W_r \text{ is } A\text{-free}\}$  is independent of  $\{W_r \leftrightarrow \rho\}$  and is implied by the event  $S = 0$  where  $S = |\{h \leq r : W_h \in A\}|$ . Hence,

$$\mathbb{E}X_r^A \geq (p(d-1))^r \mathbf{P}\left(X_{j,k}^{(h)} = 0 \text{ for all } (j, k, h), S = 0\right).$$

Since our initial vertex  $v$  was chosen uniformly at random we have  $\mathbb{E}S = \frac{r|A|}{n}$ . By Markov's inequality we get that

$$\mathbb{E}X_r^A \geq (p(d-1))^r \left[1 - \frac{r|A|}{n} - \sum_{h=2}^r \sum_{k=2}^h \sum_{j=1}^{k-1} \mathbb{E}X_{j,k}^{(h)}\right]. \quad (4.2)$$

We are left to estimate from above  $\mathbb{E}X_{j,k}^{(h)}$ . Let  $U_{j,k}$  be an independent random uniform vertex from the set

$$\left\{u \in T : |u| = k, u \text{ is a descendant of } W_j\right\},$$

and note that

$$\mathbb{E}X_{j,k}^{(h)} = (d-1)^{k-j} \mathbf{P}\left(U_{j,k} \wedge W_h = W_j, \mathcal{L}(U_{j,k}) = \mathcal{L}(W_h), U_{j,k} \leftrightarrow W_j\right).$$

The event  $\{U_{j,k} \leftrightarrow W_j\}$  is independent of the event  $\{U_{j,k} \wedge W_h = W_j, \mathcal{L}(U_{j,k}) = \mathcal{L}(W_h)\}$ , hence

$$\mathbb{E}X_{j,k}^{(h)} = (p(d-1))^{k-j} \mathbf{P}\left(U_{j,k} \wedge W_h = W_j, \mathcal{L}(U_{j,k}) = \mathcal{L}(W_h)\right). \quad (4.3)$$

Let  $U_{j,k}^1$  be the child of  $W_j$  such that  $U_{j,k}$  is a descendant of  $U_{j,k}^1$  (or if  $k = j+1$  we take  $U_{j,k}^1 = U_{j,k}$ ). For  $i, \ell \in \{1, \dots, d-1\}$  denote by  $\mathcal{A}(i, \ell)$  the event

$$\mathcal{A}(i, \ell) = \left\{W_j = w, W_{j+1} = w_i, U_{j,k}^1 = w_\ell\right\},$$

where  $w \in T$  has  $|w| = j$  and  $w_1, \dots, w_{d-1} \in T$  are the children of  $w$  in  $T$ . Note that  $U_{j,k} \wedge W_h = W_j$  if and only if  $W_{j+1} \neq U_{j,k}^1$ . Hence

$$\mathbf{P}\left(U_{j,k} \wedge W_h = W_j, \mathcal{L}(U_{j,k}) = \mathcal{L}(W_h) \mid W_j\right) = \frac{\sum_{i \neq \ell}^{d-1} \mathbf{P}\left(\mathcal{L}(U_{j,k}) = \mathcal{L}(W_h) \mid \mathcal{A}(i, \ell)\right)}{(d-1)^2} \quad (4.4)$$

Given that  $U_{j,k}^1 = w_\ell$ , we have that  $\mathcal{L}(U_{j,k})$  is distributed as the end vertex of a non-backtracking random walk of length  $k-j-1$  on  $G$  starting with the edge  $(\mathcal{L}(w), \mathcal{L}(w_\ell))$ . Similarly, given that  $W_{j+1} = w_i$ , the vertex  $\mathcal{L}(W_h)$  is distributed as the end vertex of

an independent non-backtracking random walk of length  $h - j - 1$  on  $G$  starting with the edge  $(\mathcal{L}(w), \mathcal{L}(w_i))$ . We deduce that

$$\mathbf{P}\left(\mathcal{L}(U_{j,k}) = \mathcal{L}(W_h) \mid \mathcal{A}(i, \ell)\right) = \sum_{y \in V} \mathbf{P}_{(\mathcal{L}(w), \mathcal{L}(w_\ell))}(X_{k-j-1}^{(2)} = y) \mathbf{P}_{(\mathcal{L}(w), \mathcal{L}(w_i))}(X_{h-j-1}^{(2)} = y),$$

where  $\{X_t\}$  is the non-backtracking random walk (see the notation of Section 3.1). This together with Lemma 7 implies that

$$\sum_{i \neq \ell}^{d-1} \mathbf{P}\left(\mathcal{L}(U_{j,k}) = \mathcal{L}(W_h) \mid \mathcal{A}(i, \ell)\right) \leq d(d-1) \mathbf{p}^{h+k-2j-1}(v, v).$$

This together with (4.3) and (4.4) gives that

$$\mathbb{E} X_{j,k}^{(h)} \leq \frac{d}{d-1} (p(d-1))^{k-j} \mathbf{p}^{h+k-2j-1}(v, v).$$

Putting this into (4.2) completes the proof of the lemma.  $\square$

**4.3. A second moment argument.** The following two lemmas bound from below the probability that  $B_p^A(v, r)$  is large. The two lemmas handle the cases  $\epsilon > 0$  and  $\epsilon < 0$ , since our estimates from Section 3.2 change when  $\epsilon$  changes sign. The proofs of the two are the same, except that we apply Lemma 8 in Lemma 13 and Lemma 9 in Lemma 14. To ease the reading of these lemmas, the reader is advised to think of the three significant parameters  $\epsilon, r$  and  $M$  as taking the values  $\epsilon \approx n^{-1/3}$ ,  $r \approx n^{1/3}$  and  $M \approx n^{2/3}$  (scaling window width, diameter and volume, respectively). These will be the values we will use for the proof of Theorem 1.

**Lemma 13.** *Let  $\epsilon \in (0, 1/2)$  and put  $p = \frac{1+\epsilon}{d-1}$ . Denote by  $v$  a uniform random vertex of  $G$  and let  $M$  and  $r$  be two integers satisfying*

$$\begin{aligned} (1) \quad & \frac{d}{d-1} \sum_{h=2}^r \sum_{k=2}^h \sum_{j=1}^{k-1} (1+\epsilon)^{k-j} \mathbf{p}^{h+k-2j-1}(v, v) \leq \frac{1}{2}, \\ (2) \quad & 96M < \epsilon^{-2} [(1+\epsilon)^r - (1+\epsilon)^{r/2}] (1 - e^{-\epsilon r/4}). \end{aligned}$$

*Then there exists some fixed  $c > 0$  such that if  $A \subset G$  has  $|A| \leq \frac{n}{4r}$ , then we have*

$$\mathbf{P}\left(|B_p^A(v, r)| \geq M\right) \geq c\epsilon \left(1 - (1+\epsilon)^{-r/2}\right)^2 \left(1 - e^{-\epsilon r/4}\right)^3.$$

**Proof.** For notational convenience we write  $\partial B_k$  for  $|\partial B_p^A(v, k)|$  so that  $|B_p^A(v, r)| = \sum_{k=0}^r \partial B_k$ . We have

$$\mathbf{P}\left(|B_p^A(v, r)| \geq M\right) \geq \mathbf{P}\left(\partial B_{r/2} > 0\right) \mathbf{P}\left(\sum_{k=r/2}^r \partial B_k \geq M \mid \partial B_{r/2} > 0\right).$$

By Cauchy-Schwartz we get

$$\mathbf{P}\left(\left|B_p^A(v, r)\right| \geq M\right) \geq \frac{\left[\mathbb{E}\partial B_{r/2}\right]^2}{\mathbb{E}\left[(\partial B_{r/2})^2\right]} \mathbf{P}\left(\sum_{k=r/2}^r \partial B_k \geq M \mid \partial B_{r/2} > 0\right). \quad (4.5)$$

By definition  $\mathbb{E}[\partial B_k \mid \partial B_{r/2} > 0] = \frac{\mathbb{E}\partial B_k}{\mathbf{P}(\partial B_{r/2} > 0)}$  for any  $k \geq r/2$ . Together with Proposition 11 this implies that

$$\mathbb{E}\left[\sum_{k=r/2}^r \partial B_k \mid \partial B_{r/2} > 0\right] \geq \frac{\sum_{k=r/2}^r \mathbb{E}\partial B_k}{\mathbf{P}(H_{r/2} > 0)}. \quad (4.6)$$

Lemma 12 together with assumption (1) and our assumption on  $A$  imply that for any  $k \leq r$  we have

$$\mathbb{E}\partial B_k \geq (1 + \epsilon)^k \left[\frac{1}{2} - \frac{r|A|}{n}\right] \geq \frac{(1 + \epsilon)^k}{4}. \quad (4.7)$$

We put this in (4.6) which together with (3.5) of Lemma 8 yields

$$\mathbb{E}\left[\sum_{k=r/2}^r \partial B_k \mid \partial B_{r/2} > 0\right] \geq \frac{1}{48} \epsilon^{-2} [(1 + \epsilon)^r - (1 + \epsilon)^{r/2}] (1 - e^{-\epsilon r/4}).$$

Assumption (2) and the previous display imply that

$$M < \frac{1}{2} \mathbb{E}\left[\sum_{k=r/2}^r \partial B_k \mid \partial B_{r/2} > 0\right].$$

Hence we can use the estimate  $\mathbf{P}(Z > y) \geq (\mathbb{E}Z - y)^2 / \mathbb{E}Z^2$ , valid for any non-negative random variable  $Z$  and  $y < \mathbb{E}Z$  (this estimate follows easily from Cauchy-Schwartz), and get

$$\mathbf{P}\left(\sum_{k=r/2}^r \partial B_k \geq M \mid \partial B_{r/2} > 0\right) \geq \frac{c\epsilon^{-4}[(1 + \epsilon)^r - (1 + \epsilon)^{r/2}]^2 (1 - e^{-\epsilon r/4})^2}{\mathbb{E}\left[(\sum_{k=r/2}^r \partial B_k)^2 \mid \partial B_{r/2} > 0\right]}, \quad (4.8)$$

for some  $c > 0$ . Put  $\mathcal{A} = \{\partial B_{r/2} > 0\}$  and  $\mathcal{B} = \{H_{r/2} > 0\}$  and note that Proposition 11 allows us to couple such that  $\mathcal{A} \subset \mathcal{B}$ . Since  $\mathcal{A} \subset \mathcal{B}$ , for any non-negative random variable  $X$  we have

$$\mathbb{E}[X \mid \mathcal{A}] = \frac{\mathbb{E}[X \mathbf{1}_{\mathcal{A}}]}{\mathbf{P}(\mathcal{A})} \leq \frac{\mathbf{P}(\mathcal{B})}{\mathbf{P}(\mathcal{A})} \mathbb{E}[X \mid \mathcal{B}].$$

We put  $X = (\sum_{k=0}^r \partial B_k)^2$  and use Proposition 11, which allows us to couple such that  $\partial B_k \leq H_k$ , to get

$$\mathbb{E}\left[\left(\sum_{k=r/2}^r \partial B_k\right)^2 \mid \partial B_{r/2} > 0\right] \leq \frac{\mathbf{P}(H_{r/2} > 0)}{\mathbf{P}(\partial B_{r/2} > 0)} \mathbb{E}\left[\left(\sum_{k=r/2}^r H_k\right)^2 \mid H_{r/2} > 0\right]. \quad (4.9)$$

We use Cauchy-Schwartz to bound from below the denominator and Lemma 8 to bound from above the two parts of the numerator

$$\mathbb{E}\left[\left(\sum_{k=r/2}^r \partial B_k\right)^2 \mid \partial B_{r/2} > 0\right] \leq \frac{\mathbb{E}\left[(\partial B_{r/2})^2\right]}{\left[\mathbb{E}\partial B_{r/2}\right]^2} O\left(\epsilon^{-3}(1+\epsilon)^{2r}(1-e^{\epsilon r/4})^{-1}\right).$$

We put this in (4.8) and get that

$$\mathbf{P}\left(\sum_{k=r/2}^r \partial B_k \geq M \mid \partial B_{r/2} > 0\right) \geq c \frac{\left[\mathbb{E}\partial B_{r/2}\right]^2}{\mathbb{E}\left[(\partial B_{r/2})^2\right]} \epsilon^{-1} [1 - (1+\epsilon)^{-r/2}]^2 (1 - e^{-\epsilon r/4})^3,$$

for some  $c > 0$ . We plug this into (4.5) and get

$$\mathbf{P}\left(\left|B_p^A(v, r)\right| \geq M\right) \geq c \frac{\left[\mathbb{E}\partial B_{r/2}\right]^4}{\mathbb{E}\left[(\partial B_{r/2})^2\right]^2} \epsilon^{-1} [1 - (1+\epsilon)^{-r/2}]^2 (1 - e^{-\epsilon r/4})^3. \quad (4.10)$$

By (3.4) of Lemma 8 and Proposition 11 we have

$$\mathbb{E}\left[(\partial B_{r/2})^2\right] \leq O\left(\epsilon^{-1}(1+\epsilon)^r\right),$$

and putting  $k = r/2$  in (4.7) and the result into (4.10) yields the assertion of the lemma.  $\square$

**Lemma 14.** *Let  $\epsilon \in (0, 1/2)$  and put  $p = \frac{1-\epsilon}{d-1}$ . Denote by  $v$  a uniform random vertex of  $G$  and let  $M$  and  $r$  be two integers satisfying*

$$(1) \quad \frac{d}{d-1} \sum_{h=2}^r \sum_{k=2}^h \sum_{j=1}^{k-1} (1-\epsilon)^{k-j} \mathbf{P}^{h+k-2j-1}(v, v) \leq \frac{1}{2},$$

$$(2) \quad 96M < \epsilon^{-1} r [(1-\epsilon)^{r/2} - (1-\epsilon)^r].$$

*Then there exists some fixed  $c > 0$  such that if  $A \subset G$  has  $r|A| \leq n/4$  then we have*

$$\mathbf{P}\left(\left|B_p^A(v, r)\right| \geq M\right) \geq c\epsilon^3 r^2 (1-\epsilon)^r [(1-\epsilon)^{r/2} - (1-\epsilon)^r]^2.$$

**Proof.** The proof carries on precisely as in the previous lemma up to (4.7). Instead we use assumption (1) and Lemma 12 to estimate

$$\mathbb{E}\partial B_k \geq \frac{(1-\epsilon)^k}{4}, \quad (4.11)$$

which together with (3.8) of Lemma 9 yields

$$\mathbb{E}\left[\sum_{k=r/2}^r \partial B_k \mid \partial B_{r/2} > 0\right] \geq \frac{1}{48} \epsilon^{-1} r [(1-\epsilon)^{r/2} - (1-\epsilon)^r].$$

As before, by assumption (2) we deduce that for some  $c > 0$

$$\mathbf{P}\left(\sum_{k=r/2}^r \partial B_k \geq M \mid \partial B_{r/2} > 0\right) \geq \frac{c\epsilon^{-2}r^2[(1-\epsilon)^{r/2} - (1-\epsilon)^r]^2}{\mathbb{E}\left[(\sum_{k=r/2}^r \partial B_k)^2 \mid \partial B_{r/2} > 0\right]}. \quad (4.12)$$

Inequality (4.9) still holds and we use it to estimate the denominator of the last display. As before, we use Cauchy-Schwartz to bound the denominator of (4.9) and Lemma 9 to bound the two parts of the numerator of (4.9). This gives

$$\mathbb{E}\left[(\sum_{k=r/2}^r \partial B_k)^2 \mid \partial B_{r/2} > 0\right] \leq \frac{\mathbb{E}[(\partial B_{r/2})^2]}{[\mathbb{E}\partial B_{r/2}]^2} O(\epsilon^{-3}).$$

We put this in (4.12) and the result into (4.5) to get that for some  $c > 0$  we have

$$\mathbf{P}\left(\left|B_p^A(v, r)\right| \geq M\right) \geq c\epsilon r^2 \frac{[\mathbb{E}\partial B_{r/2}]^4}{\mathbb{E}[(\partial B_{r/2})^2]^2} [(1-\epsilon)^{r/2} - (1-\epsilon)^r]^2. \quad (4.13)$$

By Proposition 11 and (3.7) of Lemma 9 we get

$$\mathbb{E}[(\partial B_{r/2})^2] \leq O(\epsilon^{-1}(1-\epsilon)^{r/2}),$$

and putting  $k = r/2$  in (4.11) and that into (4.13) gives that

$$\mathbf{P}\left(\left|B_p^A(v, r)\right| \geq M\right) \geq c\epsilon^3 r (1-\epsilon)^r [(1-\epsilon)^{r/2} - (1-\epsilon)^r]^2.$$

□

## 5. Proof of Theorems 1 and 4

To prove Theorems 1 and 4, we employ a process which explores neighborhoods of vertices in  $G_p$ . For a fixed number  $r$ , to be chosen later, the process explores neighborhoods of randomly chosen vertices up to distance  $r$ , excluding the vertices it has seen before. For the following, recall the definition of  $B_p^A(v, r)$  from Section 4. The process runs as follows. We start by choosing a uniform random vertex  $v_1$  and putting  $V_1 = B_p(v_1, r)$ . At each step  $t \geq 2$  we choose a random uniform vertex  $v_t$  of  $G$  and put  $V_t = V_{t-1} \cup B_p^{V_{t-1}}(v_t, r)$ . The process ends when we exhaust all the vertices in the graph, i.e., when  $|V_t| = n$ . For convenience we write  $V_0 = \emptyset$ .

For some fixed number  $M$  we wish to study if this process has encountered a neighborhood of size at least  $M$ . We introduce some notation. Write  $I_t$  for the indicator

random variable for the event  $\{|B_p^{V_{t-1}}(v_t, r)| > M\}$ . It is clear that if there exists  $t$  with  $I_t = 1$  then  $|\mathcal{C}_1| \geq M$ . Hence

$$\mathbf{P}\left(|\mathcal{C}_1| \geq M\right) \geq \mathbf{P}\left(\exists t \text{ with } I_t = 1\right). \quad (5.1)$$

**Lemma 15.** *Let  $p = \frac{1+\epsilon}{d-1}$  and  $T > 0$ . If  $\epsilon > 0$ , then we have*

$$\mathbb{E}|V_T| \leq \frac{2T(1+\epsilon)^{r+1}}{\epsilon},$$

and if  $\epsilon < 0$ , then we have

$$\mathbb{E}|V_T| \leq 2Tr.$$

**Proof.** We have that

$$|V_T| = \sum_{t=1}^T |B_p^{V_{t-1}}(v_t, r)|.$$

Once we condition on  $V_{t-1}$  and  $v_t$ , Proposition 11 allows us to couple such that  $|\partial B_p^{V_{t-1}}(v_t, k)| \leq H_k$ . Hence, for any  $t \leq T$  and  $\epsilon > 0$  we have

$$\mathbb{E}|B_p^{V_{t-1}}(v_t, r)| \leq \sum_{k=0}^r \mathbb{E}H_k \leq \frac{d}{d-1} \frac{(1+\epsilon)^{r+1}}{\epsilon},$$

which concludes the proof for  $\epsilon > 0$  (since  $d \geq 3$ ). The proof for  $\epsilon < 0$  goes similarly by bounding  $\mathbb{E}|B_p^{V_{t-1}}(v_t, r)| \leq 2r$ .  $\square$

**Proof of Theorem 1.** Recall that  $p = \frac{1+\lambda n^{-1/3}}{d-1}$  for some fixed  $\lambda \in \mathbb{R}$ . Since  $\mathbf{P}(|\mathcal{C}_1| \geq M)$  is increasing with  $p$  and  $\mathbf{P}(|\mathcal{C}_1| \leq M)$  is decreasing with  $p$  we may assume that  $|\lambda| \geq 1$  and the result follows for  $|\lambda| < 1$ . Theorem 1.2 of [27] (or Proposition 1 of [26]) states that for any  $\alpha > 0$  there exists  $A = A(\alpha, \lambda) > 0$  such that for *any* graph with maximum degree  $d \in [3, n-1]$  we have

$$\mathbf{P}\left(|\mathcal{C}_1| > An^{2/3}\right) \leq \alpha.$$

Thus the upper bound on  $|\mathcal{C}_1|$  implied in Theorem 1 is already proved without the need of condition (1.1).

For the lower bound, fix some small  $\delta = \delta(\lambda) > 0$  and  $\gamma = \gamma(\lambda) \in (0, 1)$ , to be chosen later. We put  $\epsilon(n) = \lambda n^{-1/3}$  and  $r = \gamma n^{1/3}$  and  $M = \delta n^{2/3}$ . Let  $h \leq r$  and  $t \leq 2h$  be two positive integers. If  $(j, k)$  is a pair of integers satisfying  $2 \leq k \leq h$  and  $1 \leq j \leq k-1$  and  $h+k-2j-1 = t$  then we must have  $h-t+1 \leq k \leq h$ . We



deduce that the number of pairs  $(j, k)$  satisfying the above requirements is at most  $t$ . Thus, we bound

$$\begin{aligned} \sum_{h=2}^r \sum_{k=2}^h \sum_{j=1}^{k-1} (p(d-1))^{k-j} \mathbf{p}^{h+k-2j-1}(v, v) &\leq e^{|\lambda|\gamma} \sum_{h=2}^r \sum_{t=1}^{2h} t \mathbf{p}^t(x, x) \\ &\leq e^{|\lambda|\gamma} \gamma n^{1/3} \sum_{t=1}^{2\gamma n^{1/3}} t \mathbf{p}^t(x, x), \end{aligned}$$

where in the first inequality we bounded  $(p(d-1))^{k-j} \leq (1 + |\lambda|n^{-1/3})^r$ . Condition (1.1) and the last display imply that we can choose  $\gamma \in (0, 1)$  small enough such that

$$\frac{d}{d-1} \sum_{h=2}^r \sum_{k=2}^h \sum_{j=1}^{k-1} (1 + \epsilon)^{k-j} \mathbf{p}^{h+k-2j-1}(v, v) \leq \frac{1}{2}.$$

For any  $\gamma > 0$  we can choose  $\delta > 0$  so small such that the two assumptions of Lemma 13 or Lemma 14 (according to whether  $\lambda > 0$  or  $\lambda < 0$ ) hold. We deduce that there exists a constant  $c = c(\lambda) > 0$  such that

$$\mathbf{P}\left(|B_p^{V_{t-1}}(v_t, r)| \geq M \mid |V_{t-1}| \leq \frac{n}{4r}\right) \geq cn^{-1/3}. \quad (5.2)$$

For some positive integer  $T$  denote by  $\mathcal{A}_T$  the event

$$\mathcal{A}_T = \left\{ \forall t \leq T \ I_t = 0 \text{ and } V_T \leq \frac{n}{4r} \right\}.$$

We prove by induction that  $\mathbf{P}(\mathcal{A}_T) \leq (1 - cn^{-1/3})^T$ . Indeed for  $T = 1$  it is obvious by (5.2). We have

$$\mathbf{P}(\mathcal{A}_T) = \mathbf{P}\left(\mathcal{A}_{T-1} \text{ and } |B_p^{V_{T-1}}(v_T, r)| < M \text{ and } |B_p^{V_{T-1}}(v_T, r)| < \frac{n}{4r} - |V_{T-1}|\right).$$

Hence we can bound

$$\mathbf{P}(\mathcal{A}_T) \leq \mathbf{P}(\mathcal{A}_{T-1}) \mathbf{P}\left(|B_p^{V_{T-1}}(v_T, r)| < M \mid \mathcal{A}_{T-1}\right) \leq \mathbf{P}(\mathcal{A}_{T-1})(1 - cn^{-1/3}),$$

where the last inequality is done by conditioning on the sets  $V_1, V_2, \dots, V_{T-1}$  and using (5.2). We now have

$$\mathbf{P}\left(\forall t \leq T \ I_t = 0\right) \leq \mathbf{P}(\mathcal{A}_T) + \mathbf{P}\left(|V_T| \geq \frac{n}{4r}\right) \leq (1 - cn^{-1/3})^T + \frac{4r\mathbb{E}V_T}{n},$$

where the last inequality follows from our estimate on  $\mathbf{P}(\mathcal{A}_T)$  and Markov's inequality. Put  $T = \alpha n^{1/3}$  for some  $\alpha > 0$  and use Lemma 15 to get

$$\mathbf{P}\left(\forall t \leq T \ I_t = 0\right) \leq e^{-c\alpha} + \frac{8\alpha\gamma e^{\lambda\gamma}}{\lambda},$$

for  $\lambda > 0$  and

$$\mathbf{P}\left(\forall t \leq T \ I_t = 0\right) \leq e^{-c\alpha} + 8\gamma^2\alpha,$$

for  $\lambda < 0$ . Putting  $\alpha = \gamma^{-1/2}$  yields that we can make both of these probabilities sufficiently small by taking  $\gamma$  small enough. This together with (5.1) concludes the proof.  $\square$

**Proof of Theorem 4.** We proceed similarly to the proof of Theorem 1. Take  $r$  defined in Theorem 4. Let  $(h, t_1, t_2)$  be a triple of positive integers satisfying

$$h \leq r, \quad t_1 \leq 2h, \quad t_2 \leq h \wedge t_1.$$

The number of pairs  $(j, k)$  satisfying  $h + k - 2j - 1 = t_1$  and  $k - j = t_2$  is at most 1. Therefore we can bound

$$\begin{aligned} \sum_{h=2}^r \sum_{k=2}^h \sum_{j=1}^{k-1} (1+\epsilon)^{k-j} \mathbf{p}^{h+k-2j-1}(v, v) &\leq \sum_{h=2}^r \sum_{t_1=1}^{2h} \sum_{t_2=1}^{h \wedge t_1} (1+\epsilon)^{t_2} \mathbf{p}^{t_1}(v, v) \\ &\leq \epsilon^{-1} r \sum_{t=1}^{2r} [(1+\epsilon)^{t \wedge r} - 1] \mathbf{p}^t(v, v). \end{aligned}$$

Thus, condition (1.3) implies that for large enough  $n$  we have

$$\frac{d}{d-1} \sum_{h=2}^r \sum_{k=2}^h \sum_{j=1}^{k-1} (1+\epsilon)^{k-j} \mathbf{p}^{h+k-2j-1}(v, v) \leq \frac{1}{2}.$$

Now put  $M = \frac{\delta n \epsilon}{\log^3(n \epsilon^3)}$  and observe that if  $\delta > 0$  is small enough, our choices of  $M$  and  $r$  satisfy the two assumptions of Lemma 13. Hence, there exists some constant  $c > 0$  such that

$$\mathbf{P}\left(|B_p^{V_{t-1}}(v_t, r)| \geq M \mid |V_{t-1}| < \frac{n}{4r}\right) \geq c\epsilon. \quad (5.3)$$

We proceed similarly to before. For some positive integer  $T$  denote by  $\mathcal{A}_T$  the event

$$\mathcal{A}_T = \left\{ \forall t \leq T \ I_t = 0 \text{ and } V_T \leq \frac{n}{4r} \right\}.$$

We prove by induction as before that  $\mathbf{P}(\mathcal{A}_T) \leq (1 - c\epsilon)^T$  and deduce, using Lemma 15 as before, that

$$\mathbf{P}\left(\forall t \leq T \ I_t = 0\right) \leq (1 - c\epsilon)^T + \frac{8Tr(1+\epsilon)^r}{\epsilon n} \leq e^{-c\epsilon T} + \frac{8T\epsilon}{\log^2(n\epsilon^3)}.$$

Taking  $T = \epsilon^{-1} \log(n\epsilon^3)$  makes this probability tend to 0 and (5.1) concludes the proof.  $\square$

## 6. Concluding remarks and open problems

1. The proof of Theorem 4 shows in fact that condition (1.3) could be replaced by the slightly weaker condition

$$\limsup_n \epsilon^{-1} r \sum_{t=1}^{2r} [(1 + \epsilon)^{t \wedge r} - 1] \mathbf{p}^t(v, v) < \limsup_n \frac{d(n) - 1}{d(n)}. \quad (6.1)$$

We have not found, however, examples in which this condition holds and (1.3) does not.

2. In the case that  $G$  is the hamming  $m$ -cube  $\{0, 1\}^m$ , an upper bound on the size of the scaling window of order  $n^{-1/\log^{2/3}(n)}$  is given in [9]. It is broadly believed (and conjectured in [9]) that the scaling window of the hypercube is of order  $\Theta(n^{-1/3})$ , however, this is still wide open.
3. The case of the high-dimension discrete torus  $[m]^d$  (for large fixed  $d$  and  $m \rightarrow \infty$ ) seems even harder. In this case there is no sub-constant upper bound on the size of the scaling-window.
4. Another problem is to derive the existence of a scaling window for expanders of low girth, where the critical probability is not  $\frac{1}{d-1}$ . In particular, we conjecture that the conclusions of Theorem 2 hold for *any* expander family, only around a different  $p_c$ , larger than  $\frac{1}{d-1}$ .

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